Nonlinear Dynamics and Chaos

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The present experiment deals with the fundamental properties of nonlinear dynamical systems, chaotic behaviour and chaos. In the experiment two different kinds of nonlinear oscillators are investigated, viz. the inverted pendulum and a Shinriki oscillator. To find chaotic behaviour and periodic orbits the phase portrait, autocorrelation function as well as the Fourier transform of the measured signal are calculated. In case of the Shinriki oscillator a Feigenbaum diagram and a phase diagram of the control parameter is recorded.

BASICS

The following section is a short introduction to nonlinear dynamical systems, chaos and their mathematical description.

Dynamical Systems

For many physical problems it is possible to rewrite the equations of motion in a set of first order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{F}(\boldsymbol{x}(t), t), \tag{1}$$

where t is the time, $\boldsymbol{x}(t)$ the trajectory in real space and $\boldsymbol{F}(\boldsymbol{x}(t), t)$ a vector field. Note that $\boldsymbol{F}(\boldsymbol{x}(t), t)$ is in general a smooth function. The whole time evolution of a dynamical system is determined by the initial conditions and hence it is always possible to find a deterministic solution. A more formal definition is:

Definition 1 (from [1]). *A smooth dynamical system on $\mathbb{R}^n$ is a continuously differentiable function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, where $\phi(t, \boldsymbol{x}) = \phi_t(\boldsymbol{x})$ satisfies*

1. *$\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the identity function: $\phi_0(\boldsymbol{x}_0) = \boldsymbol{x}_0$.*
2. *The composition $\phi_t \circ \phi_s = \phi_{t+s}$ for each $t, s \in \mathbb{R}$.*

Here $\phi_t(\boldsymbol{x})$ is the so called evolution function or flow which describes how the system in the configuration $\boldsymbol{x}$ evolves in time t. If $\boldsymbol{F}(\boldsymbol{x}(t))$ does not explicitly depend on t we deal with a so called autonomous systems. Calculation of an analytical solution in case of nonlinear systems is rarely feasible. A physical system is called nonlinear if additional time dependent variables of higher orders appear in equation (1). Typical examples of nonlinear systems are damped driven pendulums, the three body problem or the Navier-Stokes equation [1, 2].

Phase Space: All possible states of a dynamical system are represented in the phase space which consists of all conceivable values of space and momentum variables and is therefore a vector space. A more formal representation is given by the phase space vector

$$\boldsymbol{\xi} = \left(q^1, \dots, q^f, p_1, \dots, p_f\right)^T, \tag{2}$$

where q^i are the positions, p_i the corresponding momenta and f the number of degrees of freedom [3].

Dissipative Systems: In the case of Hamiltonian systems the phase space distribution function is constant along the trajectories of the system according to Liouville's theorem, i.e. the phase space volume is preserved. If the systems contains additional dissipative terms the phase space volume decreases and the system is called dissipative. A typical example is a damped harmonic oscillator.

Trajectories in the phase space must not intersect, otherwise the intersection point leads to indefinite time evolution. Note that nonlinear dynamical systems may exhibit chaotic behaviour, which is in general not equivalent to chaos.

Chaos

Dynamical systems that are highly sensitive to initial conditions may contain chaotic behaviour or chaos, i.e. small differences in the initial conditions results in different outcomes. Therefore a long term prediction is in general impossible. Although these systems are deterministic it is not possible to predict their future. This behaviour is called deterministic chaos. Typical examples are the three body problem or the Lorenz attractor.

Lyapunov Exponent: To describe the evolution behaviour of a dynamical system it is common to define the Lyapunov exponent

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{\|\delta x^i(t)\|}{\|\delta x_0^i\|}\right), \tag{3}$$

where the index i describes the spatial direction, $\|\delta x^i(t)\|$ the distance between the i-th component of the observed curve and the reference curve at time t and $\|\delta x_0^i\|$ the distance at time 0. The Lyapunov exponent is a measure for the rate of separation of infinitesimally close trajectories. For $\lambda > 0$ the trajectories diverge, for $\lambda < 0$ they converge. The distance remains constant if $\lambda = 0$.

Attractor: An attractor describes the long time evolution of a dynamical system for a wide variety of initial conditions, i.e. a set of points in phase space towards the system contracts. Once an attractor is reached it is impossible to leave it. An attractor can be a single point (or a finite set of non-continuously distributed points), a curve (limit cycle), a manifold (torus) or a strange attractor like the Lorenz attractor.

Signal Analysis and Autocorrelation

This section shows how to identify important properties of a dynamical system by Fourier transform the measured signal, looking at the autocorrelation function or the power spectral density. Henceforth the notation $x(t)$ is used for the measured signal.

Fourier Transformation: The Fourier transformation of a function $f(t)$ is given by

$$\mathcal{F}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t. \tag{4}$$

The inverse Fourier transformation is

$$f(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(\omega) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega. \tag{5}$$

Note that $f(t)$ is an integrable function.

Autocorrelation: The autocorrelation function $\gamma(\tau)$ is defined by

$$\begin{aligned} \gamma(\tau) &\equiv \langle x(t), x(t+\tau) \rangle & (6) \\ &= \int_{-\infty}^{\infty} x(\tau) x(t+\tau) \, \mathrm{d}\tau & (7) \end{aligned}$$

and is a tool for finding repeating pattern like periodic behaviour. In case of chaotic behaviour the autocorrelation function is zero. For periodic behaviour one observes an autocorrelation function much larger than zero. Note that a measurement process itself is finite in time and therefore it is not possible to integrate from $-\infty$ to ∞ and the total energy can not reach a finite value. Is this the case it is helpful to use the time average

$$\bar{\gamma}(\tau) \equiv \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t+\tau) x(t) \, \mathrm{d}t, \tag{8}$$

where T is the period time.

Power Spectral Density: The power spectral density (PSD) describes the optical power per frequency interval and has the physical dimension $[\mathrm{PSD}] = \mathrm{W\,Hz^{-1}}$. We can also say it describes how the power of signal or time series are disturbed over different frequencies. It is defined by

$$S_{xx} \equiv \int_{-\infty}^{\infty} \gamma(\tau) \mathrm{e}^{-\mathrm{i}\omega\tau} \, \mathrm{d}\tau, \tag{9}$$

where $\gamma(\tau) = \langle x(t), x(t+\tau) \rangle$ is the autocorrelation function of the measured signal. The PSD is a statistical measure which can be calculated by averaging over many measurement results.

Discret Dynamical Systems

Sometimes it is possible to reduce a differential equation to an iterated function. Due to the lower dimensional space a easier visualisation is possible. Moreover instead of integrating it is now possible to solve the problem by iterating the function over and over. An iterative function in one dimension is given by the map

$$u_i \mapsto u_{i+1} = f(u_i). \tag{10}$$

A point is called a fixed point if $u(x_0) = x_0$. A periodic point of period n is given by $u^n(x_0) = x_0$ for some $n > 0$. As a consequence a periodic orbit repeats itself. A periodic point x_0 has minimal period n if n is the least positive integer for which $u^n(x_0) = x_0$ [1, 2].

Logistic Map: A typical example of a discrete dynamical system is the logistic map

$$x_{n+1} = r x_n (1 - x_n), \tag{11}$$

where $r > 0$ is a control parameter and $x \in [0, 1]$. The logistic map is a mathematical model for a driven damped oscillator. Plotting x over r for many iterations one has the so called Feigenbaum diagram, cf. figure 1 which allows the determination of the Feigenbaum constants δ and α. The constant δ is given by the ratio of the length of two following periodic windows

$$\delta = \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \approx 4.6692. \tag{12}$$

The constant α is given by the ratio of the width of two following periodic windows.

$$\alpha \approx 2.5029. \tag{13}$$

ANALYSIS

In this section we discuss the experimental procedures and the analysis of the measured data.

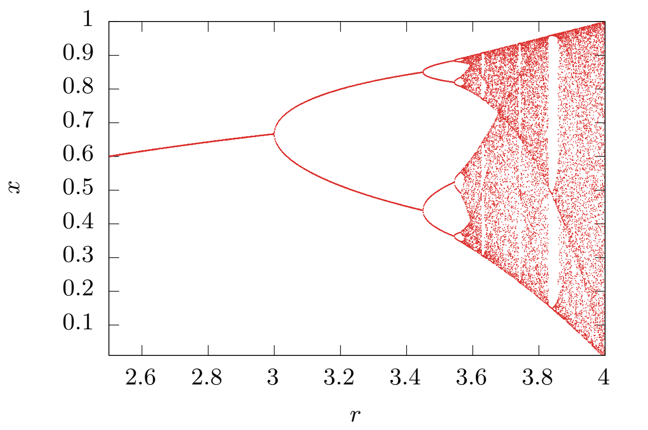

FIG. 1. Feigenbaum diagram. The logistic map for many iterations is shown.

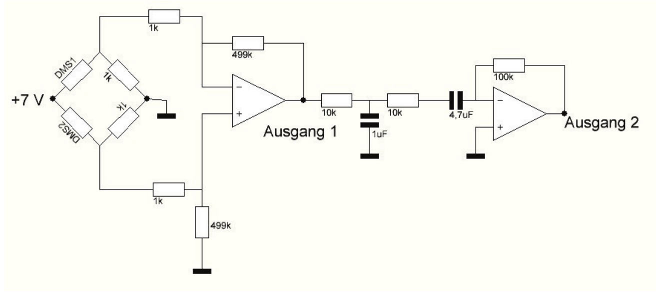

FIG. 2. Circuit diagram of the inverted pendulum.

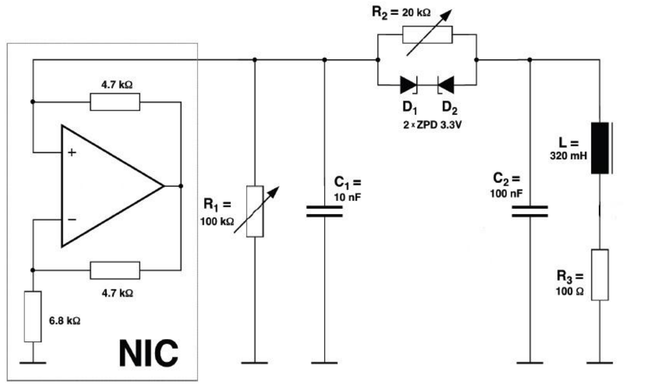

FIG. 3. Circuit diagram of a Shinriki oscillator. R_1 and R_2 are the control parameters.

Experimental Task and Setup

The experiment is mainly divided into four experimental tasks.

Inverted Pendulum: With help of a computer software the displacement and velocity are measured as a function of time to investigate the phase diagram and the Fourier transform of the displacement. Measurements were done for different driving frequencies. The aim is to observe orbits with period one, two or higher, as well as chaotic behaviour.

Shinriki-Oscillator: Like in the previous task measurements were done with help of a computer software to observe different vibrational states. This time the resistances R_1 and R_2 are the control parameters. It is only allowed to vary these parameters.

Feigenbaum diagram: For creating a Feigenbaum diagram it is necessary to do a lot of measurements for varying control parameter R_1. By plotting just the maxima of the measured data, i.e. the voltage as function of the control parameter R_1 a wild Feigenbaum diagram appears. With help of the diagram it is then possible to determine the Feigenbaum constant δ.

Phase diagram: With help of the Shinriki oscillator it is possible to create a phase diagram of the two control parameters R_1 and R_2. Therefore the measurement had to be done by varying one of the control parameters while the other one is held constant and vice versa. While one of the control parameters is varied the transitions between different vibrational states are noted down.

The experimental setup of the inverted pendulum is depicted in figure 2 and the Shinriki oscillator in figure 3. The data were recorded with LabVIEW.

Inverted Pendulum

The following experimental task deals with the inverted pendulum. Before it is possible to record measurements the experimental setup had to be adjusted. We chose for the oscillating mass two small copper discs and two small metal discs. The springs were applied at an altitude of 7 cm. All subsequent measurements were done with this adjustment.

By varying the driving frequency it was possible to observe different vibrational sates. To identify them the autocorrelation function (8) and the Fourier transform (4) of the measured signal were calculated with `python`. For present adjustment it was possible to identify the following vibrational states:

- An orbit with period three as shown in figure 4 (a). With help of the Fourier transformed signal this identification is possible. Obviously there are three resonance frequencies besides the driving frequency $\nu = 0.7$ Hz. There are also higher orders visible in the figure.
- An Orbit with period one as depicted in figure 4 (b). This corresponds to the high resonance peak near the driving frequency 0.80 Hz. This can also be seen in the decreasing autocorrelation function.

- An orbit with period two as depicted in figure 4 (c). In the autocorrelation function it is possible to observe two decreasing oscillations. The two resonance frequencies are also visible in the Fourier transformed signal.

One can see that it is possible to identify the resonance frequencies in the Fourier transformed figures. The autocorrelation function decays for all measurements to zero. As discussed in the basics this is due to the finite measurement time.

Shinriki Oscillator

Like in the previous experimental task we are now interested in observing different vibrational states. Instead of varying the driving frequency now the two control resistances R_1 and R_2 are varied. For the measurements shown in figure 7 the resistance $R_2 = 18.6\,\mathrm{k\Omega}$ while R_1 is varied. The measurements were done with a LabVIEW program. The analysis is the same as in the previous task. The measured data and the corresponding autocorrelation functions as well as the Fourier transforms are depicted in figure 7.

With help of the figure it is possible to identify the following vibrational states:

- An orbit with period one is depicted in figure 7 (a). The autocorrelation function decreases continually and no other oscillations are superimposed.

- An orbit with period two is shown in figure 7 (b) and (f). The autocorrelation functions shows this. The Fourier transformed signal shows two peaks corresponding to the resonance frequencies.

- An orbit with period three is depicted in figure 7 (e). All three resonances are visible in the Fourier transformed signal.

- An orbit with period four is given in 7 (c). The Fourier transformed signal shows all four peaks where two of them are dominant compared to the others.

- Mono-scroll chaos was observed in figure 7 (d). The Fourier transformed signal shows the chaotic behaviour. One dominant peak surrounded by noise is visible.

- Double-scroll chaos is depicted in figure 7 (g). Obviously the autocorrelation function shows that there is no correlation in the signal. The two peaks in the Fourier transformed signal are surrounded by noise.

Feigenbaum Diagram

For the Shinriki oscillator it is possible to observe a Feigenbaum diagram by doing a lot of measurements. Therefore we choose the control parameter $R_2 = 9.3\,\mathrm{k\Omega}$ and varied R_1. For each varied R_1 the measurement is recorded. By searching the maximum of the recorded data a data point to the adjusted R_1 for the Feigenbaum diagram is found. Plotting all these data one has the Feigenbaum diagram depicted in figure 5. To determine the Feigenbaum constant δ it is necessary to find the data points which correspond to a bifurcation point. A numerical search yields

$$R_{1\to 2} = 16.80\,\mathrm{k\Omega}, \tag{14}$$

$$R_{2\to 4} = 17.62\,\mathrm{k\Omega}, \tag{15}$$

$$R_{4\to 8} = 17.80\,\mathrm{k\Omega}. \tag{16}$$

The subscript in $R_{i\to j}$ indicates that for this control parameter a change from period i to period j occurs. Note that the figure shows only an interval of R_1 where a valid analysis is possible. With the bifurcation points the determination of the Feigenbaum constant is possible. For the measured data we find

$$\delta = \frac{R_{2\to 4} - R_{1\to 2}}{R_{4\to 8} - R_{2\to 4}} = 4.56. \tag{17}$$

The literature value is $\delta = 4.6692$.

Phase Diagram

To record a phase diagram of the control parameters R_1 and R_2 it is necessary to hold first one of the control parameter constant while the other one is varied. We held R_2 fixed and varied then R_1 from $0\,\mathrm{k\Omega}$ to $100\,\mathrm{k\Omega}$ and noted down the points where a change of the vibrational state is observed. Then a new value for R_2 is chosen and the measurement process is repeated. The measured data were then plotted in a phase diagram to identify the parameter constellations were different vibrational states are possible. The diagram is depicted in figure 6. The figure shows that for high values of R_2 it is possible to end at a great orbit with period one. Once this orbit is reached it is impossible to go backwards, i.e. we end on an attractor. Also for small values of R_2 it is only possible to observe an orbit of period one. Nevertheless, as depicted for greater values of R_2, it is possible to observe nearly all vibrational states discussed in figure 7. Note that for some cases it was due to the finite precision of the potentiometer a bit annoying to identify the correct orbits. Moreover the figure shows that double-scroll chaos and mono-scroll chaos appear very often.

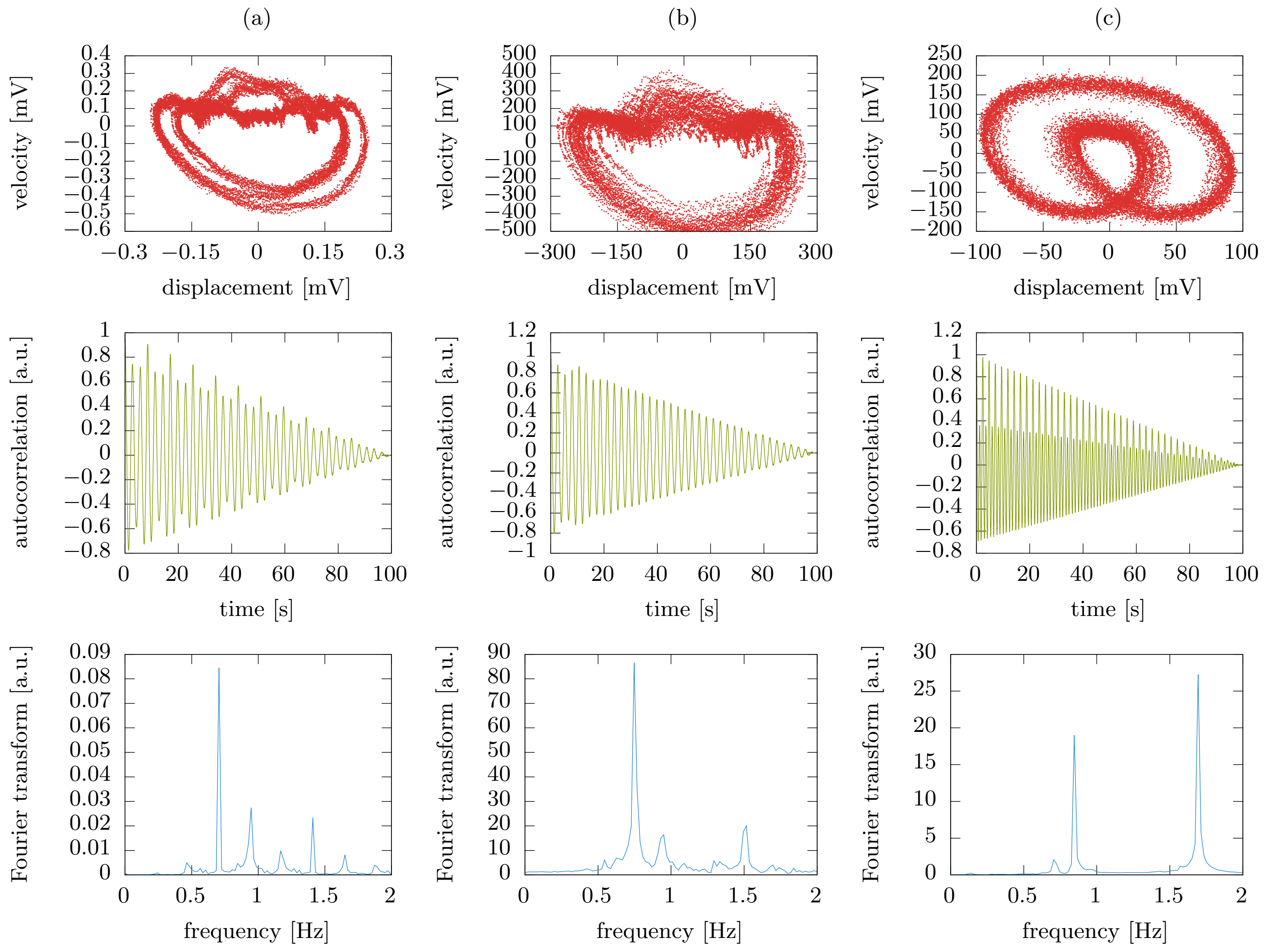

FIG. 4. From top to bottom: (a) Phase diagram, autocorrelation function and Fast Fourier transform of the measured signals with a driving frequency $\nu = 0.70\,\mathrm{Hz}$. Depicted is an orbit with period three. (b) Phase diagram, autocorrelation function and Fast Fourier transform of the measured signals with a driving frequency $\nu = 0.80\,\mathrm{Hz}$. Shown is an orbit with period one. (c) Phase diagram, autocorrelation function and Fast Fourier transform of the measured signals with a driving frequency $\nu = 0.85\,\mathrm{Hz}$. Depicted is an orbit with period two.

ERROR DISCUSSION

Due to the finite precision of the measurement devices and adjustment instruments, e.g. the potentiometers it is useful to calculate the propagation of uncertainty for the determined Feigenbaum constant. The calculation of $\Delta\delta$ is given by

$$\Delta\delta = \sum_i \left| \frac{\partial \delta}{\partial x_i} \right| \Delta x_i, \tag{18}$$

and gives the total error. The relative errors Δx_i of the quantities x_i are given by the precision of the measurement instruments. In our case x_i corresponds to the bifurcation values $R_{i\to j}$ in equation (17). For all three bifurcation points we can assume an relative error of $\Delta R_{i\to j} = 0.05\,\mathrm{k\Omega}$ which corresponds to the precision of the used potentiometers. We then find

$$\Delta\delta = 3.09. \tag{19}$$

The Feigenbaum constant is therefore given by

$$\delta = 4.56 \pm 3.09. \tag{20}$$

Compared with the literature value $\delta = 4.6692$ we are within the total error limits. Furthermore the relative error is about 2.34 %, which is nice.

SUMMARY

In the experimental task the different vibrational states of an inverted pendulum were recorded and the Fourier transform as well as the autocorrelation function determined and depicted in figure 4. As shown in the figure

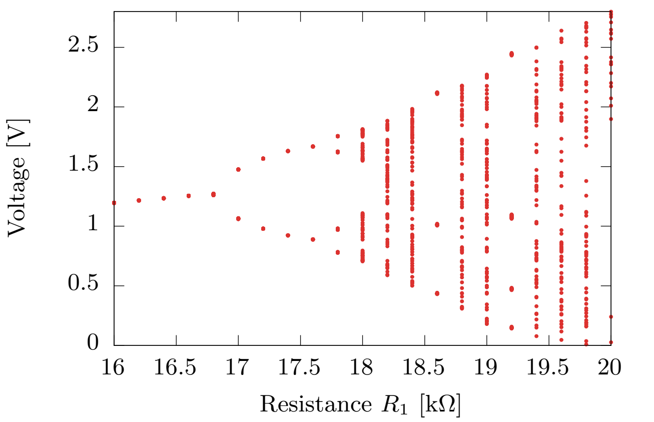

FIG. 5. Feigenbaum diagram of the Shinriki oscillator for fixed $R_2 = 9.3\,\mathrm{k\Omega}$ and varying R_1.

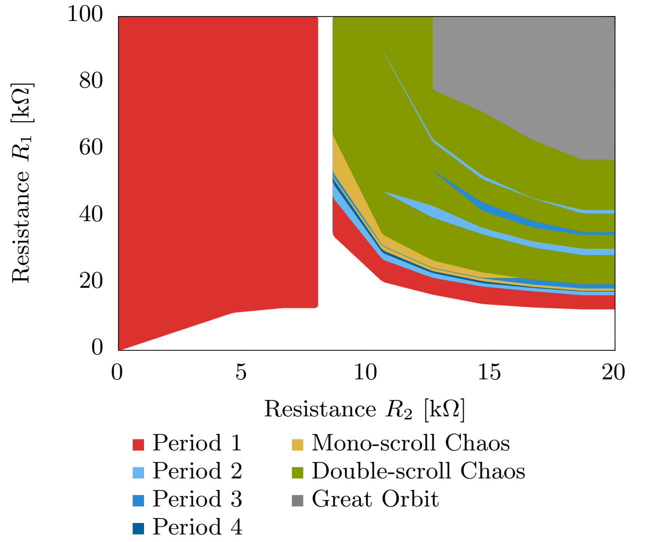

FIG. 6. Phase diagram of the Shinriki oscillator. Depicted is R_1 as function of R_2. The areas where different vibrational states are possible are coloured.

it was possible to investigate orbits of period one, two and three. The resonance peaks are given in the Fourier transformed signal.

The same observations were done in the case of the Shinriki oscillator as depicted in figure 7. Moreover it was possible to create a Feigenbaum diagram 5 and determine the Feigenbaum constant to

$$\delta = 4.56 \pm 3.09. \tag{21}$$

Compared to the literature value $\delta = 4.6692$ this corresponds to a relative error of 2.34 %. Moreover it was possible to observe a phase diagram 6 to identify the different areas of vibrational states as function of the two control parameters R_1 and R_2.

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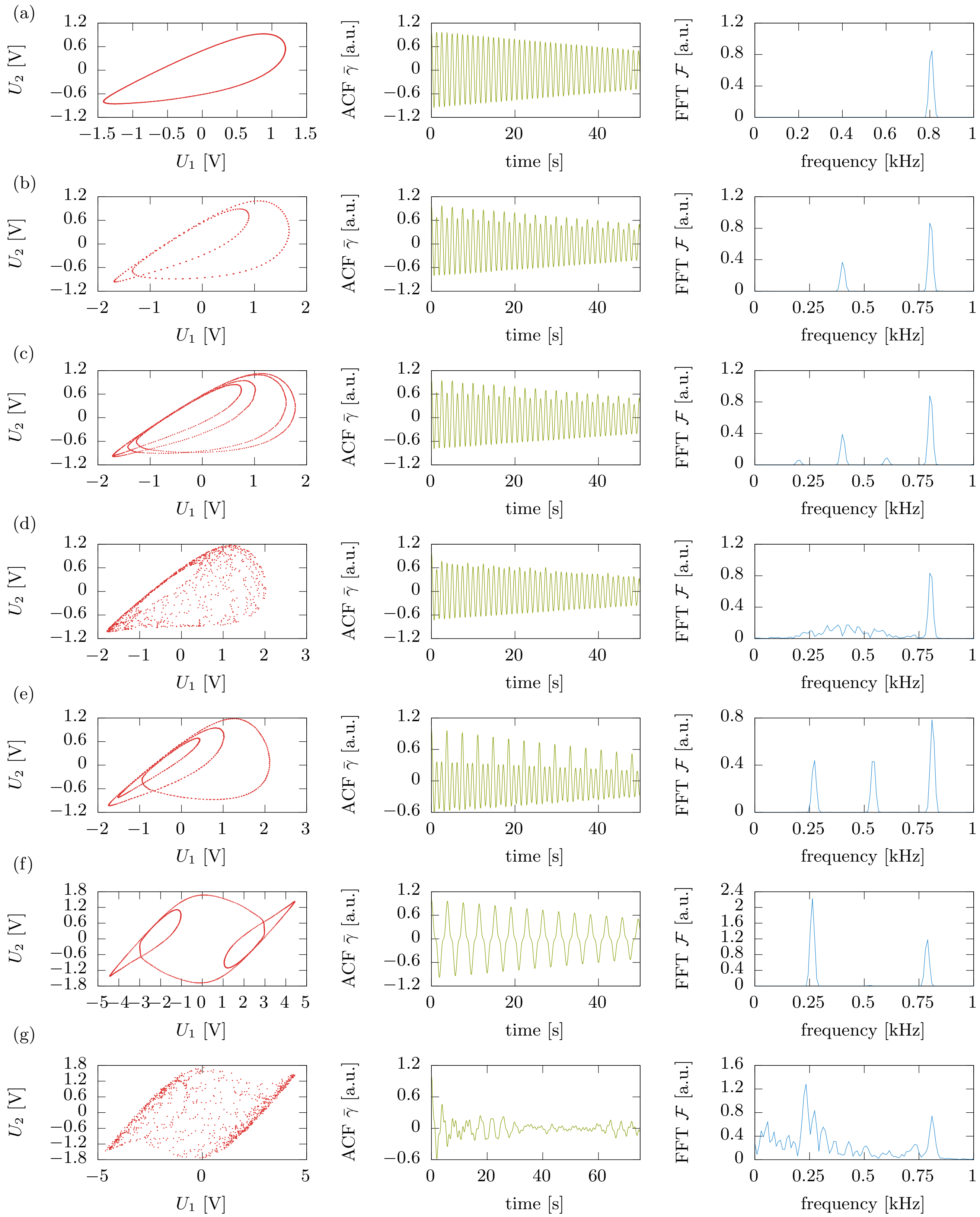

FIG. 7. From left to right. Phase diagram, autocorrelation function and Fourier transform for: (a) Orbit of period one with $R_1 = 16.0\,\mathrm{k\Omega}$, $R_2 = 18.6\,\mathrm{k\Omega}$. (b) Orbit of period two with $R_1 = 17.5\,\mathrm{k\Omega}$, $R_2 = 18.6\,\mathrm{k\Omega}$. (c) Orbit of period four with $R_1 = 17.8\,\mathrm{k\Omega}$, $R_2 = 18.6\,\mathrm{k\Omega}$. (d) Mono- scroll chaos for $R_1 = 18.5\,\mathrm{k\Omega}$, $R_2 = 18.6\,\mathrm{k\Omega}$. (e) Orbit of period three with $R_1 = 18.6\,\mathrm{k\Omega}$, $R_2 = 18.6\,\mathrm{k\Omega}$. (f) Orbit of period two with $R_1 = 28.8\,\mathrm{k\Omega}$, $R_2 = 18.6\,\mathrm{k\Omega}$. (g) Double-scroll chaos for $R_1 = 30.6\,\mathrm{k\Omega}$, $R_2 = 18.6\,\mathrm{k\Omega}$.